

A characterization of linear admissible transformations for the m -Travelling Salesmen Problem

Xavier BERENGUER

Universidad Politécnic de Barcelona, Centro de Cálculo,
Barcelona, Spain

Received May 1977

Revised July 1978

Some methods for solving the Multiple-Travelling Salesmen type of problems are based on certain transformations of the distances matrix. The transformed matrix entails a new objective function, for which an optimal solution is sought.

This paper deals with these transformations and in it those which are valid in a linear context are analyzed. "Linear admissible transformations" are studied, with their characterization, and the more relevant transformations given by other authors when solving the Single-Travelling Salesman Problem and the Delivery Problem are discussed under this approach.

1. Introduction

The m -Travelling Salesmen Problem (m -TSP) can be stated as follows: There are m salesmen who must visit a set of n clients. Each salesman tour starts and ends at the central office (named here 0). Each client must be visited only once by one salesman. A set of m tours must be found so that the sum of the involved distances is minimized.

The m -Travelling Salesmen Problem includes a wide variety of subproblems. For instance, the case in which $m = 1$ corresponds to the Single Travelling Salesman Problem. The case in which additional constraints are applied, such as limited capacity of a vehicle associated to each salesman and unknown variable m , is referred to as the Delivery Problem.

The author would like to thank the referees for their interest and helpful comments which have improved the readability of this paper. In particular, the basic conclusion of Section 5 was contributed by one of the referees.

There are a great number of algorithms which solve these problems. In some cases, special transformations of the distances matrix are applied, either as a simplifying rule, or as a basis for a whole procedure. For instance, in the Clarke and Wright algorithm [1] to solve the Delivery Problem, the "savings" a_{ij} are calculated as:

$$a_{ij} = d_{i0} + d_{0j} - d_{ij}, \quad i, j = 0, 1, 2, \dots, n, \quad d_{ij}: \text{elements of distances matrix } D$$

and, after having arranged the arcs in a decreasing order according to their savings, the algorithm proceeds with a progressive selection of arcs following heuristic rules yielding finally a solution to the problem.

Other transformations of the distances matrix, like saving, can be implemented, so that when the same selection procedure is applied, other solutions (better or worse) are obtained. However, not any transformation will be valid and care must be taken that the quality ordering of solutions is the same after and before the transformation.

This paper deals with the validity of this type of transformations of the distances matrix.

2. Admissible mapping, admissible transformation and admissible matrix

Consider a m -TSP with an associated distances matrix D and a set of solutions \mathcal{S} . A particular solution shall be represented by a square matrix S of order $n + 1$

$$\begin{array}{c} \text{to} \\ 0 \ 1 \ 2 \ \dots \ n \\ \text{from} \\ \vdots \\ n \end{array} \begin{array}{|c|} \hline x \\ \hline x \\ \hline x \\ \hline \dots \\ \hline x \\ \hline \end{array}$$

with elements $s_{ij} \in \{0, 1\}$, depending on whether arc ij is included (1) or not (0) in the solution, and null diagonal. According to the specifications of the m -TSP

$$\sum_{i=0}^n s_{i0} = \sum_{j=0}^n s_{0j} = m,$$

$$\sum_{i=0}^n s_{ii} = 0$$

and for $j = 1, 2, \dots, n$,

$$\sum_{i=0}^n s_{ij} = \sum_{i=0}^n s_{ji} = 1$$

the objective of the problem being:

$$[\text{MIN}] \sum_{i,j} s_{ij} d_{ij}.$$

The "total distance computation" of a solution is a mapping F from \mathcal{S} to \mathbf{R} (real numbers set), $\mathcal{S} \xrightarrow{F} \mathbf{R}$ which defines a total order relation of \mathcal{S} on \mathbf{R} , such as

$$S_1 \succcurlyeq S_2 \succcurlyeq \dots \succcurlyeq S_*, \quad S_i \in \mathcal{S}$$

when

$$F(S_1) \geq F(S_2) \geq \dots \geq F(S_*)$$

where S_* is an optimal solution in a minimizing context. The referred total order relation arranges \mathcal{S} according to a "quality" criterion.

A mapping $\tau, \mathcal{S} \xrightarrow{\tau} \mathbf{R}$ will be called *admissible mapping* if and only if

$$\tau(S_1) \geq \tau(S_2) \geq \dots \geq \tau(S_*)$$

or

$$\tau(S_1) \leq \tau(S_2) \leq \dots \leq \tau(S_*).$$

In the first case it will be said that τ is an admissible mapping in the *direct sense*; in the second that it is one of *inverse sense*.

Clearly, for a mapping τ to be admissible a necessary and sufficient condition is that either

$$\tau(S_i) - \tau(S_j) \geq 0 \quad \forall S_i, S_j \in \mathcal{S}, \text{ if } S_i \succ S_j$$

or

$$\tau(S_i) - \tau(S_j) \leq 0 \quad \forall S_i, S_j \in \mathcal{S}, \text{ if } S_i \succ S_j$$

In order to define the admissible transformations and matrices it is convenient to introduce the operations \otimes and \lceil :

– operation \otimes (element by element product) between

two matrices of identical order

$$C = A \otimes B \quad \text{where } c_{ij} = a_{ij} \cdot b_{ij};$$

– operation \lceil (sum of all elements of a matrix)

$$\lceil(B) = \beta \quad \beta = \sum_i \sum_j b_{ij}.$$

(Note that the mapping F "total distance computation" of a solution can be expressed as $\lceil(S \otimes D)$).

The matrix $t(D)$ associated to such an admissible mapping (where the value induced at \mathbf{R} for each solution could be expressed as $\lceil(S \otimes t(D))$) will be called admissible matrix, and the operation t through which this new matrix is obtained from D , will be called *admissible transformation* (term of Vo-Khac [5]).

It is obvious that if

$$\lceil(S \otimes t(D)) = \delta \cdot \lceil(S \otimes D) + \alpha \quad \forall S \in \mathcal{S}, \delta, \alpha \in \mathbf{R}, (1)$$

then t is an admissible transformation. The admissible transformations which satisfy the last expression will be called *linear admissible transformations* (LAT), to which *linear admissible matrices* (LAM) are associated.

For LAT, the direct/inverse sense concept explained before can be immediately deduced from δ . When $\delta > 0$ the LAT is in the direct sense; when $\delta < 0$ the LAT is in the inverse sense.

3. The vector space of linear admissible matrices

Suppose that a m -TSP has the particular solutions $S_i \in \mathcal{S}$ ordered as

$$S_1 \succcurlyeq S_2 \succcurlyeq \dots \succcurlyeq S_*$$

because $\lceil(S_1 \otimes D) \geq \lceil(S_2 \otimes D) \geq \dots \geq \lceil(S_* \otimes D)$.

The set \mathcal{M} of all the LAM for this problem is defined as follows:

$$\begin{aligned} \mathcal{M} = \{ & M^+ \mid \lceil(S_1 \otimes M^+) \geq \lceil(S_2 \otimes M^+) \\ & \geq \dots \geq \lceil(S_* \otimes M^+) \} \\ & \cup \{ M^- \mid \lceil(S_1 \otimes M^-) \leq \lceil(S_2 \otimes M^-) \\ & \leq \dots \leq \lceil(S_* \otimes M^-) \} \end{aligned}$$

(M^+ : LAM of direct sense; M^- : LAM of inverse sense).

The structure of \mathcal{M} will now be presented.

Two fundamental operations must be considered: addition and homotethy. It is easy to demonstrate that the addition (in the traditional concept of matrix addition) of two LAM yields with another LAM.

Besides, the operation *homotethy*, defined as

$$M' = \lambda M \quad \text{with } m'_{ij} = \lambda m_{ij}, \lambda \in \mathbb{R}, M \in \mathcal{M}$$

obviously yields another LAM M' .

According to the properties of these operations it can be seen that $(\mathcal{M}, +)$ is a vector space on \mathbb{R} with the homotethy as a law of external composition. It is therefore a subspace of the vector space of matrices of order $(n+1)(n+1)$ with null diagonal, associated to the operations addition and homotethy.

Our attention is turned now to calculating the dimension of vector space \mathcal{M} .

Theorem. *The rank of the system*

$$\begin{cases} (S \otimes M) = \delta [(S \otimes D) + \alpha] \\ \delta, \alpha \in \mathbb{R}, S \in \mathcal{S}, M \in \mathcal{M}, \\ \text{is } \leq n(n-1) \end{cases}$$

See demonstration in Appendix.

Theorem. *The dimension of the vector space \mathcal{M} is $\geq 2n+2$.*

The dimension ρ of \mathcal{M} will be precisely the number of degrees of freedom of the system

$$(S \otimes M) = \delta [(S \otimes D) + \alpha]$$

thus, $\rho = n(n+1) + 2 - c$, being c the rank of this system. From the previous theorem $c \leq n(n-1)$, therefore

$$\rho \geq 2n+2.$$

While the number of solutions $\text{Card}(\mathcal{S})$ is $\geq n(n-1)$ the dimension of \mathcal{M} is $2n+2$. But if $\text{Card}(\mathcal{S}) \leq n(n-1)$ the dimension increases as this number of solutions diminishes. Thus, for a problem with a single solution, any matrix will obviously be admissible (be it linear or not).

Corollary. *If D is symmetrical the dimension of \mathcal{M} is*

$$\geq \frac{1}{2}n(n+1) + 2 - n(n-1), \text{ i.e. } \geq n+2.$$

A base for \mathcal{M} will be now defined.

The matrices Ω_i^R , Ω_i^C and Ω_0 are defined in the following way:

For $\Omega \in \mathbb{R}$ and $i, j = 0, 1, 2, \dots, n$

Ω_i^R is a $(n+1)(n+1)$ matrix with elements

$$\omega_{ij} = 0 \text{ for } i \neq l \text{ and } \omega_{ij} = \Omega:$$

Ω_i^C is a $(n+1)(n+1)$ matrix with elements

$$\omega_{ij} = 0 \text{ for } j \neq l \text{ and } \omega_{ij} = \Omega;$$

Ω_0 is a $(n+1)(n+1)$ matrix with elements

$$\omega_{ij} = 0 \text{ for } i \neq 0 \text{ and } j \neq 0, \text{ and } \omega_{0j} = \omega_{i0} = \Omega.$$

It is easy to see that matrices Ω_i^R , Ω_i^C and Ω_0 are LAM, because for $S \in \mathcal{S}$

$$[(S \otimes \Omega_i^R) = [(S \otimes \Omega_i^C) = \Omega, \quad i \neq 0.$$

$$[(S \otimes \Omega_0) = 2m\Omega$$

The following can be taken as a base of the vector space \mathcal{M} :

$$\{D, \Omega_0, \Omega_1^R, \Omega_1^C, \Omega_2^R, \Omega_2^C, \dots, \Omega_n^R, \Omega_n^C\}$$

(Ω arbitrary for each matrix).

It can be proved easily that they are linearly independent matrices.

Thus, any LAM can be expressed as a linear combination of basic matrices. So, if $M \in \mathcal{M}$

$$M = K_D D + K_0 \Omega_0 + K_1^R \Omega_1^R + \dots + K_n^C \Omega_n^C,$$

then

$$\begin{aligned} [(S \otimes M) &= [(S \otimes (K_D D + K_0 \Omega_0 + \dots + K_n^C \Omega_n^C)) \\ &= K_D [(S \otimes D) + K_0 [(S \otimes \Omega_0) \\ &\quad + \dots + K_n^C [(S \otimes \Omega_n^C). \end{aligned} \quad (2)$$

In order to adjust this eq. (2) with the linear expression (1) of Section 2, the following Ω values will be taken: $\Omega = (1/2m)$ for Ω_0 and $\Omega = 1$ for the rest of basic matrices, obtaining the normalized base

$$\left\{ D, \left(\frac{1}{2m} \right)_0, 1_1^R, 1_1^C, 1_2^R, 1_2^C, \dots, 1_n^R, 1_n^C \right\}.$$

So in the linear expression (1) coefficients δ and α can be interpreted as follows:

– Coefficient δ is the component of the LAM with respect to the basic matrix D .

– Coefficient α is the sum of components of the LAM with respect to the basic matrices $(1/2m)_0, 1_1^R, 1_1^C, \dots, 1_n^C$.

If D is symmetrical, the base can be simplified by combining the matrices Ω_i^R, Ω_i^C , thus obtaining matrices Ω_i of order $(n+1)(n+1)$ with elements $\omega_{ij} = 0$ for $i \neq l$ and $j \neq l$ and $\omega_{ij} = \omega_{il} = \Omega, i, j = 0, 1, 2, \dots, n$.

The normalized base for the symmetrical case is then

$$\left\{ D, \left(\frac{1}{2m} \right)_0, 1_1, 1_2, \dots, 1_n \right\}.$$

In order to characterize a LAM it will be sufficient to indicate its components with respect to the base.

$$[K_D \ K_0 \ K_1^R \ \dots \ K_n^C] .$$

The Linear Admissible Transformation concept can be now precised as follows: A LAT t is a linear transformation which operated on the vector D of \mathcal{M} giving another vector of \mathcal{M}

$$[K_D \ K_0 \ K_1^R \ \dots \ K_n^C]' = t[1 \ 0 \ 0 \ \dots \ 0]' . \quad (3)$$

So, every LAT can be expressed by means of the left handside of this last equation.

4. Some particular linear admissible transformations

We are now in a position to interpret some possible LAT which appear in practice when solving m -TSP or more specific subproblems such as the single-TSP and the Delivery Problem.

Basic linear admissible transformations

Vo-Khac [5] defines two "basic" LAT: homotethy and transfer.

The *homotethy* λ is the LAT $[\lambda \ 0 \ 0 \ \dots \ 0]$.

Transfer θ consists in the addition of a constant θ to the row and column of D corresponding to a point i ($i = 0, 1, 2, \dots, n$).

It is the LAT $[1 \ 2m\theta \ 0 \ \dots \ 0]$ when $i = 0$ or the LAT $[1 \ 0 \ 0 \ \dots \ 0 \ \theta \ \theta \ 0 \ \dots \ 0]$ when $i \neq 0$.

It is useful to define two additional basic LAT: row-transfer and column-transfer.

Row-transfer θ consists in the addition of a constant θ to the row of D corresponding to a point i ($i = 0, 1, 2, \dots, n$).

It is the LAT

$$[1 \ m\theta \ -\frac{1}{2}\theta \ \frac{1}{2}\theta \ -\frac{1}{2}\theta \ \frac{1}{2}\theta \ \dots \ -\frac{1}{2}\theta \ \frac{1}{2}\theta] \quad \text{when } i = 0$$

or the LAT

$$[1 \ 0 \ 0 \ \dots \ 0 \ \theta \ 0 \ \dots \ 0] \quad \text{when } i \neq 0.$$

Column-transfer θ consists in the addition of a constant θ to the column of D corresponding to a point i ($i = 0, 1, 2, \dots, n$).

It is the LAT

$$[1 \ m\theta \ \frac{1}{2}\theta \ -\frac{1}{2}\theta \ \frac{1}{2}\theta \ -\frac{1}{2}\theta \ \dots \ \frac{1}{2}\theta \ -\frac{1}{2}\theta] \quad \text{when } i = 0$$

or the LAT

$$[1 \ 0 \ 0 \ \dots \ 0 \ 0 \ \theta \ 0 \ \dots \ 0] \quad \text{when } i \neq 0.$$

The reduction of the distances matrix

In a Branch and Bound method for the solution of the Single-TSP, Little et al. [3] propose a rule for obtaining a lower bound called "reduction" of the distances matrix. It is easy to see that it is a LAT.

Let

$$g_i = \min_j d_{ij}$$

for each row i . The reduction at $i = 0$ is the LAT

$$[0 \ -g_0 \ \frac{1}{2}g_0 \ -\frac{1}{2}g_0 \ \dots \ \frac{1}{2}g_0 \ -\frac{1}{2}g_0]$$

and the reduction at i ($i = 1, 2, \dots, n$) is the LAT

$$[0 \ 0 \ \dots \ 0 \ -g_i \ 0 \ \dots \ 0] .$$

Thus, while considering D , the reduction by rows is the LAT

$$M = [1 \ -g_0 \ (-g_1 + \frac{1}{2}g_0) \ -\frac{1}{2}g_0 \ (-g_2 + \frac{1}{2}g_0) \ \dots \\ (-g_n + \frac{1}{2}g_0) \ -\frac{1}{2}g_0] .$$

When this LAT is applied on D , a new matrix D' is obtained. Let

$$h_j = \min_i d'_{ij}$$

for each column j . In a similar way the other LAT can be specified as:

$$N = [0 \ -h_0 \ -\frac{1}{2}h_0 \ (-h_1 + \frac{1}{2}h_0) \ -\frac{1}{2}h_0 \ \dots \ -\frac{1}{2}h_0 \\ (-h_n + \frac{1}{2}h_0)] .$$

Therefore, the reduction rule in Little et al. algorithm is the LAT $M + N$, that is:

$$[1 \ (-g_0 - h_0) \ (-g_1 - \frac{1}{2}h_0 + \frac{1}{2}g_0) \ (-h_1 + \frac{1}{2}h_0 - \frac{1}{2}g_0) \\ \dots \ (-g_n - \frac{1}{2}h_0 + \frac{1}{2}g_0) \ (-h_n + \frac{1}{2}h_0 - \frac{1}{2}g_0)]$$

Saving

For solving the Delivery Problem, Clarke and Wright [1] define the saving a_{ij} , mentioned in Section 1, $a_{ij} = d_{i0} + d_{0j} - a_{ij}$ which is the LAT

$$[-1 \ 0 \ d_{10} \ d_{01} \ d_{20} \ d_{02} \ \dots \ d_{n0} \ d_{0n}] .$$

For symmetrical problems, to which usually the Clarke and Wright algorithm is applied, the LAT is

$$[-1 \ 0 \ 2d_{01} \ 2d_{02} \ \dots \ 2d_{0n}]$$

Note that it can be interpreted as the sum of n transfers $2d_{0i}$ plus the homotethy $(-n - 1)$.

π transformation of Gaskell

Again for solving the symmetrical delivery problem, Gaskell [2] defines the transformation

$$\pi_{ij} = a_{ij} - d_{ij} \quad (a_{ij} \text{ saving})$$

which is the LAT

$$[-2 \ 0 \ 2d_{01} \ 2d_{02} \ \dots \ 2d_{0n}] .$$

or the sum of saving and the homotethy -1 .

λ transformation of Gaskell

Gaskell [2] defines also the transformation

$$\lambda_{ij} = a_{ij}(\bar{d} + |d_{0i} - d_{0j}| - d_{ij}) ,$$

$$\bar{d} = \frac{\sum_{i=1}^n d_{0i}}{n}, \quad a_{ij} \text{ saving.}$$

It cannot be said that this is a LAT, because the product of two LAM is not necessarily another LAM. Moreover, it cannot be said either that it is an admissible transformation of another kind. It is easy to design some numerical example where the λ transformation yields a different quality-hierarchy of its solutions. Hence, the λ transformation of Gaskell is not an admissible transformation.

“Affinité pondérée”

Vo-Khac [5] proposes the “affinité pondérée” g_{ij} :

$$g_{ij} = \gamma(i) + \gamma(j) + m(d_{0i} + d_{0j}) - (n + m - 2) d_{ij}$$

if $i, j \neq 0$,

$$g_{0i} = \gamma(i) - (n - 2) d_{0i} \quad \text{with } \gamma(i) = \sum_{\substack{k \neq 0 \\ k \neq i}} d_{ik} .$$

It is a LAT resulting from the sum of n transfers $(2\gamma(i) + 2md_{0i})$ for $i = 1, 2, \dots, n$ and the homotethy $(-n - m + 2)$, i.e. the LAM

$$[-(n + m - 2) \ 0 \ 2(\gamma(1) + md_{01}) \ \dots \ 2(\gamma(n) + md_{0n})] .$$

ψ transformation on ICL–VSP program

In a package developed by ICL (Vehicle Scheduling Program Mark I, II, III, quoted in NCC [4]), the fol-

lowing transformation is used:

$$\psi_{ij} = \pi_{ij}(d_{0i} + d_{0j}) - d_{ij} , \quad \pi: \text{Gaskell's transformation.}$$

As product $\pi_{ij}(d_{0i} + d_{0j})$ does not yield necessarily a LAM, ψ transformation is not a LAT.

5. Conclusions

A characterization of linear admissible transformations on the m -TSP distances matrix has been presented. The results obtained can be summarized as follows:

For the m -Travelling Salesmen Problem, the only linear admissible transformations of the distances matrix D are those obtained by adding constants θ_i and θ_j to row i and column j of λD ($i, j = 0, 1, 2, \dots, n$; λ constant).

On this basis, some transformations found in the literature for solving m -TS subproblems (Single-Travelling Salesman Problem and Delivery Problem) have been analyzed, finding some erroneous ones.

Other linear transformations should be discussed on the same theoretical basis, before any other consideration such as efficiency, practical results obtained, etc., could be made.

Appendix

Theorem. *The rank of the system*

$$[(S \otimes M) = \delta [(S \otimes D) + \alpha], \quad \delta, \alpha \in \mathbb{R}, S \in \mathcal{S}, M \in \mathcal{M}$$

is $\leq n(n - 1)$.

The matrix of the corresponding homogeneous system with unknown m_{ij} ($i, j = 0, 1, 2, \dots, n$), α and δ has ω rows, where $\omega = \text{Card}(\mathcal{S})$ and $n(n + 1) + 2$ columns (see Fig. 1).

The elements of this matrix have value 0 or 1. As the $n(n + 1)$ first elements of each row are precisely all the elements of a solution to the problem, it shall be convenient to represent them as s_{ij}^k , where k indicates the row (or solution) referred to and ij indicates the column, as an element of the representation S of a solution.

The last two elements of the system matrix will be called s_α^k and s_δ^k where k indicates the row.

Through the following considerations, it is possible to eliminate dependent columns in this system matrix.

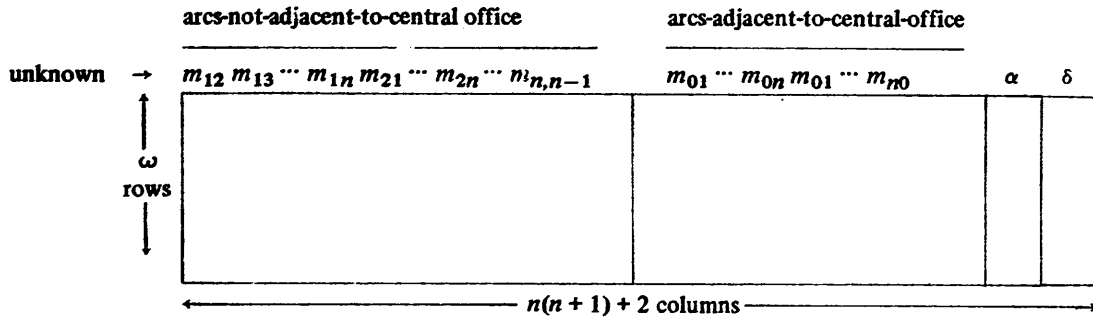


Fig. 1. System matrix.

- The elements of column α are all -1 , that is, a linear combination of elements s_{ij}^k

$$s_{\alpha}^k = -1 = \frac{-1}{n-m} \sum_{i,j=1}^n s_{ij}^k$$

because in every solution there are $n-m$ arcs-not-adjacent-to-central-office.

- The elements of column δ are a linear combination of elements s_{ij}^k

$$s_{\delta}^k = - \sum_{i,j=0}^n d_{ij} s_{ij}^k$$

because each element of this column is the real distance of each solution, with changed sign.

- The elements of columns m_{0j} ($j = 1, 2, \dots, n$) can also be expressed as a linear combination of elements s_{ij}^k

$$s_{0l}^k = \frac{1}{n-m} \sum_{i,j=1}^n s_{ij}^k - \sum_{i=1}^n s_{il}^k, \quad l = 1, 2, \dots, n$$

because for every solution a certain point l is reached only once and if it has not been reached through arcs-not-adjacent-to-central-office (there are $n-m$ of them), then it must happen that the point l has been reached from the central office.

- The elements of columns m_{j0} ($j = 1, 2, \dots, n$) are also a linear combination of elements s_{ij}^k

$$s_{l0}^k = \frac{1}{n-m} \sum_{i,j=1}^n s_{ij}^k - \sum_{j=1}^n s_{lj}^k, \quad l = 1, 2, \dots, n$$

because for every solution a certain point l is left only once and if it has not been left through arcs-not-adjacent-to-central-office it is because the exit from l has been made to central office.

Therefore, only the columns of arcs-not-adjacent-

to-central-office must be considered when calculating the rank of the system (Fig. 2).

It will be proved that the rank for this reduced matrix is $n(n-1)$, whenever $\omega \geq n(n-1)$.

Given the relationship

$$\sum_{i,j=1}^n \mu_{ij}^k s_{ij}^k = 0$$

the aim is to prove that it must be $\mu_{ij}^k = 0, i, j = 1, 2, \dots, n$ and $k = 1, 2, \dots, \omega$.

As dependence of adjacent-to-central-office-arcs has been already proved, for demonstration purposes the most unfavourable case must be considered, that is when $m = 1$ and a solution is a hamiltonian path between the n clients.

Assume that a certain solution k is represented as a sequence of clients

$$i_2 \ i_3 \ \dots \ i_{n-1} \ i_n$$

and take also the solutions obtained with its circular permutations:

$$\begin{aligned} & i_1 \ i_2 \ \dots \ i_n \ i_1 \\ & i_3 \ i_4 \ \dots \ i_1 \ i_2 \\ & \vdots \\ & i_n \ i_1 \ \dots \ i_2 \ i_{n-1} . \end{aligned}$$

For the first solution it is clear that

$$\mu_{i_1 i_2}^k + \mu_{i_2 i_3}^k + \dots + \mu_{i_{n-1} i_n}^k = 0 ,$$

for the second

$$\mu_{i_2 i_3}^k + \dots + \mu_{i_n i_1}^k = 0 ,$$

and so for the last one

$$\mu_{i_n i_1}^k + \dots + \mu_{i_{n-2} i_{n-1}}^k = 0 .$$

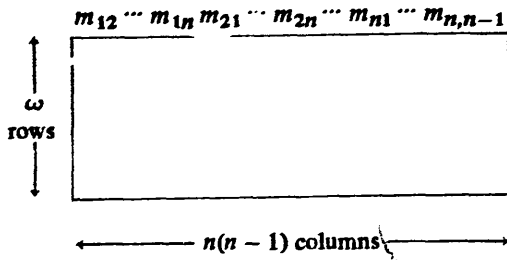


Fig. 2. System reduced matrix.

Adding all equations it results

$$\mu_{i_1 i_2}^k + \mu_{i_2 i_3}^k + \dots + \mu_{i_{n-1} i_n}^k + \mu_{i_n i_1}^k = 0.$$

Subtracting it from the former equations it can be obtained finally

$$\mu_{i_1 i_2}^k = \mu_{i_2 i_3}^k = \dots = \mu_{i_{n-1} i_n}^k = \mu_{i_n i_1}^k = 0.$$

As the method serves for any solution, it shall be possible to prove that $\mu_{ij}^k = 0$ for $k = 1, 2, \dots, \omega, i, j = 1, 2, \dots, n$.

Thus the columns of arcs-not-adjacent-to-central-office are independent, and therefore it has been proved that the rank of the system is $n(n-1)$ whenever $\text{Card}(\mathcal{C}) \geq n(n-1)$. When $\text{Card}(\mathcal{C}) < n(n-1)$ the rank of the system will obviously be $\text{Card}(\mathcal{C})$.

References

- [1] G. Clarke and J.W. Wright, Scheduling of vehicles from a central depot to a number of delivery points. *Operations Res.* 11 (1963) 568.
- [2] T.J. Gaskell, Bases for vehicle fleet scheduling. *Operations Res. Quart.* 18 (1967) 281.
- [3] J.D.C. Little, K.G. Murty, D.W. Sweeney and C. Karel, An algorithm for the travelling salesman problem. *Operations Res.* 11 (1963) 979.
- [4] National Computer Center, Computers in vehicle scheduling, NCC, Manchester (1969).
- [5] Vo-Khac, La régularisation dans les problèmes combinatoires et son application dans les problèmes de tournées. *Rev. Française Automat. Informat. Recherche Opérationnelle* 2 (1971) 59.