COMPUTING A K-INDEPENDENT SET OF MAXIMAL WEIGHT ON A PARTIALLY ORDERED SET: A RESEARCH CASE HISTORY

X. BERENGUER

This is a tutorial paper presenting the research carried out on the Sperner-Erdös problem, that is the problem of computing a Maximal Weighted -Independent Set on a Partially Ordered set. Results are shown in the same order as the research was made: analysis and solution to the Sperner (sub)problem (K=1) and generalisation of this result yielding a polynomial solution to the Sperner-Erdös problem.

1. INTRODUCTION

It is a well known result the "Sperner's --- theorem" which states that in the lattice of all subsets of an n-set the maximal number - of incomparable (i.e. not related) elements is $\binom{n}{[\frac{n}{2}]}$. It arises in a natural way the ---- "Sperner problem", that is the problem of -- computing the maximal number of incomparable elements in any partially ordered set.

Erdös gave a generalisation of Sperner's -- theorem stating that the maximal cardinality of any set of incomparable elements which -- has no more than k members lying on any chain is

$$\sum_{\ell=0}^{k-1} \binom{n}{\left[\frac{n+\ell}{2}\right]}$$

For any partially ordered set the correspon- ding problem that arises will be referred to as the "Sperner-Erdös problem".

Both problems can be formulated while assig- ning non constant weights for each element - of the poset. These more general versions of the problem will be studied here.

The paper presents the real course of the -- research carried out for solving the Sperner -Erdös problem. After the statement of this problem, the related subproblems are studied yielding a solution to the Sperner-problem. The aim of the procedure for this solution is then generalized and the Sperner-Erdös -- problem is solved too.

The final results of this research are reported elsewhere (/1/ and /2/), emphasis is --- made here on the intermediate steps between them and their connection. The author will be happy if some light on the "methodology" of the research is extracted from this work; this intention is to clarity -somehow- the obscureness of the way to get the so called "final results", which in general don't in-- clude the experience behind them anyway.

2. THE SPERNER-ERDÖS PROBLEM

Given a poset P, a weight function ω: $P \rightarrow R^+$ (such that for $A \subseteq P$, $\omega(A) = \sum_{x \in A} \omega(x)$) and a - number $k \in Z^+$, a <u>k-chain</u> is a linearly orde-- red set of k elements, and a <u>k-independent set</u> I_k (or a k-Sperner set) is a subset of P with no (k+1)-chains, the Sperner-Erdös -- problem is to compute

$$\max_{I_k \subseteq P} \omega(I_k)$$

3. THE SPERNER PROBLEM

A first relaxation for the Sperner-Erdös Problem can be made fixing k=1. In that case the Sperner Problem is faced. The 2-chains are the usual adges and teh problem is to - compute the Maximal Weighted Independent set on a poset.

Other relaxation can be made when considering uniform weights. The problem is then to ---

X. Berenguer, Facultat d'Informàtica, Jordi Girona, 31 - Barcelona-34

find the independent set of Maximal Cardinality. For a general graph this problem is - known to be NP-Complete, but is polynomially solvable for particular cases (bipartite --- graphs, when applying the König - Egervary - theorem, and others, see Garey-Johnson /5/). For posets, Golumbic /7/ gives a polynomial algorithm.

A Vertex Cover of a graph G(V,E) is a subset $C \subseteq V$ such that for all part $(x,y) \in E$, either $x \in C$ or $y \in C$ or both $x,y \in C$.

It is easy to show that given an Independet Set I, then C = V-I is a vertex cover, so:

$$\max_{I \subseteq V} \omega(I) = \sum_{x \in V} \omega(x) - \min_{C \subseteq V} \omega(C)$$

From this equation, the problem of computing a Maximal weighted Independent set on any -- graph is equivalent to the problem of computing a Minimal weighted vertex cover.

Given a weighted poset P, define P^2, the --- 2-Copy of P, as follows:

$$P^2 = \{(x,i) : x \in P \text{ and } i \in \{0,1\},\ \text{arcs } (x,i) < (y,j) \text{ iff } x<y \text{ and } j>i,\ \text{weights } \omega'(x,i) = \omega(x)\}$$

For a subset of vertices A in P define the - "corresponding subset" A' in P^2 as $A' = A_0 \cup A_1$, where

$$A_0 = \{(x,0 \mid x \in A \text{ and } \exists\ y \in V-A \text{ such that } x>y, y \notin A\}$$

$$A_1 = \{(x,1 \mid x \in A \text{ and } \exists\ z \in V-A \text{ such that } x<z, z \notin A\}$$

For a subset of vertices A' in P^2 define the corresponding subset A in P as follows:

$$A = \{x \mid (x,i) \in A'\}$$

Lemma: A subset C is a Minimal Weighted Vertex Cover in P if its corresponding subset C in P^2 is a Minimal Weighted Vertex Cover in P^2, and viceversa.

As a more general result is presented below. the proof of this lemma is not given here. - The key fact in that proof is the transitivity property of the poset (see /1/).

Hence, the Sperner problem on P can be reduced to the problem of computing a Minimal -- Weighted Vertex Cover on P^2.

P^2 is a bipartite graph. For any bipartite graph $B = (V^A, V^B, E)$ the problem of computing a Minimal Weighted Vertex Cover can be formulated as the following Integer Linear - Program:

$$\text{MIN } \omega_i x_i \qquad i = 1,2,\ldots n, \qquad n = |V^A| + |V^B|$$

$$(1) \begin{cases} x_i + x_j \geq 1 & (i,j) \in E \\ x_i \in \{0,1\} \end{cases}$$

The matrix of this program is totally unimodular because it is the transposed matrix of the incidence matrix of a bipartite graph, - which is known to be totally unimodular (see /6/). The extreme points of the convex ---- polyedron will be integers and so an equivalent Linear Program with integer solutions - can be formulated. The famous Kachian's result, giving a polynomial algorithm for the Linear Programming Problem yields a polyno-- mial solution to our problem.

However a more efficient procedure can be obtained. The dual of (1) for our case will be

$$\text{MAX} \sum_{(i,j) \in E} \theta_{ij} - \sum_{1}^{n} y_k$$

$$\begin{cases} \sum_{j \in \Gamma^+(i)} \theta_{ij} - y_i \leq \omega_i & \forall i \in V^A, \ (\Gamma^+ \text{ successor fun.}) \\ \sum_{i \in \Gamma^-(j)} \theta_{ij} - y_j \leq \omega_j & \forall j \in V^B \ (\Gamma^- \text{ predecessor fun.}) \\ \theta_{ij} \geq 0, \ y_k \geq 0 \end{cases}$$

Making $y_k = 0$, $k = 1,2,\ldots n$, the infinite -- "scaled" solutions are avoided, and the resulting program is:

$$\text{MAX} \sum_{(i,j) \in E} \theta_{ij}$$

$$\begin{cases} z_i - \sum_{j \in \Gamma^+(i)} \theta_{ij} = 0 & \forall i \in V^A \\ \sum_{i \in \Gamma^-(j)} \theta_{ij} - z_j = 0 & \forall j \in V^B \\ \theta_{ij} \geq 0 \\ 0 \leq z_i \leq \omega_i \end{cases}$$

Now we are faced with a Linear Program which corresponds to a Max-Flow Problem, and efficient algorithms are known for this problem (for instance, Dinic-Karzanov /3/ running in $\Theta(n^3)$ time complexity, or Galil /4/ running in $(n^{2.33})$ time complexity).

Applying all the previous results we get the following equations which solve the Sperner Problem:

$$\max_{I\subseteq P} \omega(I) = \sum_{x\in P} \omega(x) - \min_{C\subseteq P} \omega(C) =$$

$$= \sum_{x\in P} \omega(x) - \min_{C\subseteq P^2} \omega(C) =$$

$$= \sum_{x\in P} \omega(x) - \text{MAX FLOW } (P^2)$$

The following algorithm sums up the procedure:

a) Construct P^2

b) Solve MAX FLOW in P^2. That will yield a MIN CUT C' on P^2, the corresponding set C in P will be a Minimal Weighted Vertex Cover in P.

c) I = P - C and I is the desired Maximal Weighted Independent Set.

4. COMING BACK TO THE SPERNER-ERDÖS PROBLEM

Recall that the main problem was to find $\max_{I_k\subseteq P} \omega(I_k)$, where I_k is a Maximal Weighted k-Independent set (a subset of P with no (k+1)-chains).

As for the above subproblem, first we transform the MAX problem to a MIN problem. A set $C_k \subseteq P$ is a <u>k-chain cover</u> if it contains at least one member of every chain in P. It is easy to show then that

$$\max_{I_k\subseteq P} \omega(I_k) = \sum_{x\in P}\omega(x) - \min_{C_{k+1}\subseteq P} \omega(C_{k+1})$$

We define de multipartite graph P^k, the k-copy of P as follows

$$P^k = \{(x,i) = x\in P \text{ and } 0\le i<k,$$
$$\text{ares } (x,i)<(y,j) \text{ iff } x<y \text{ and } j=i+1,$$
$$\text{weights } \omega'(x,i)=\omega(x)\}$$

The following lemma transforms the problem of computing a Minimal Weighted (k+1)-chain cover in P to the one of computing a Minimal Weighted (k+1)-chain cover in $P^{(k+1)}$.

<u>Lemma</u>: $\min_{C_{k+1} \text{ in } P} \omega(C_{k+1}) = \min_{C'_{k+1} \text{ in } P^{k+1}} \omega'(C'_{k+1})$

<u>Proof</u>: Let $\mathcal{C}_{k+1}$ be the set of all (k+1)-chain covers, First we shall prove $\mathcal{C}_{k+1}(P^{k+1}) \subseteq \mathcal{C}_{k+1}(P)$ and secondly $\mathcal{C}_{k+1}(P) \subseteq \mathcal{C}_{k+1}(P^{k+1})$ and therefore the lemma will be proved.

1) $\mathcal{C}_{k+1}(P^{k+1}) \subseteq \mathcal{C}_{k+1}(P)$

Define the correspondence ρ:

$\mathcal{C}_{k+1}(P^{k+1}) \rightarrow \mathcal{C}_{k+1}(P)$:

$\rho(C') = \{x:(x,i)\in C' \text{ for some } i\}$

We shall show that $\rho(C') \subseteq \mathcal{C}_{k+1}(P)$.

That is easy: Suppose a chain in P^{k+1}:

$(x_0,0)<(x_1,1)< \dots\dots\dots <(x_k,k)$

Clearly there is an i such that $(x_i,i)\in C'$ (as C' covers every chain in P^{k+1}). But then, by the definition of ρ, $x_i \in \rho(C')$, and therefore

$\rho(C') \subseteq \mathcal{C}_{k+1}(P)$ and $\mathcal{C}_{k+1}(P^{k+1}) \subseteq \mathcal{C}_{k+1}(P)$

and

$$\min_{C\in\mathcal{C}_{k+1}(P)} \omega(C) \le \min_{C'\in\mathcal{C}_{k+1}(P^{k+1})} \omega'(C')$$

2) $\mathcal{C}_{k+1}(P) \subseteq \mathcal{C}_{k+1}(P^{k+1})$

Define the correspondence $\varphi: \mathcal{C}_{k+1}(P) \rightarrow \mathcal{C}_{k+1}(P^{k+1})$
For a (k+1)-chain cover $C\in\mathcal{C}_{k+1}(P)$ and for all $x\in P$, let $\ell_C(x)$ be the length of the longest chain in the (k-Independent Set) P - C which leads up (but does not include) x.

$$\varphi(C) = \{(x,\ell_C(x)) : x\in C\}$$

We shall show that $\varphi(C) \subseteq \mathcal{C}_{k+1}(P^{k+1})$

For any chain in $P: L = \{x_0,x_1,\dots,x_k\}$ we must show that there exist some $i_0 \in \mathbb{Z}^+$, $0\le i_0 \le k$ such that $x_{i_0} \in C$, $C\in \mathcal{C}_{k+1}(P)$ and $\ell_C(x_{i_0})=i_0$, then $(x_{i_0},i_0)\in\varphi(C)$ and so the the corresponding chain in P $:L' = \{(x_0,0),(x_1,1),\dots,(x_k,k)\}$ will be covered.

Let us observe the behavior of $\ell_C(x)$ function. It is easy to see that

i) $\ell_C(x_0) \geq 0$

ii) For all i, $\ell_C(x_{i+1}) \geq \ell_C(x_i)$ and $\ell_C(x_{i+1}) = \ell_C(x_i)$ implies $x_i \in C$.

iii) $\ell_C(x_k) \leq k$ and $\ell_C(x_k) = k$ implies $x \in C$.

Therefore $\ell_C(x)$ is an increasing discrete function defined into the interval 0 (or more), k (or less). See Fig. 1 for a graphical interpretation of this fact:

A continous analog of $\ell_C(x_i)=i$ is drawn (in heavy line), and three examples of $\ell_C(x_i)$:

(1) which crosses $\ell_C(x_i)=i$ in $i=k$

(2) which crosses it at some intermediate point in $|0,k|$.

(3) which crosses it at i=0.

Clearly $\ell_C(x_i)$ meets $\ell_C(x_i)=i$ at some point and so the set $\{i:\ell_C(x_i)=i\}$ is non empty.

Recall we are trying to see that there exist some i_0 such that $x_{i_0} \in C$, $C \in \mathcal{C}_{k+1}(P)$ and $\ell_C(x_{i_0}) = i_0$. Now it is easy to find this i_0.

a) If $\ell_C(x_k)=k$, take $i_0=k$ (and obviously $x_k \in C$).

b) If $\ell_C(x_k) < k$, take $i_0 = \mathrm{MAX}\{i:\ell_C(x_i)=i\}$,

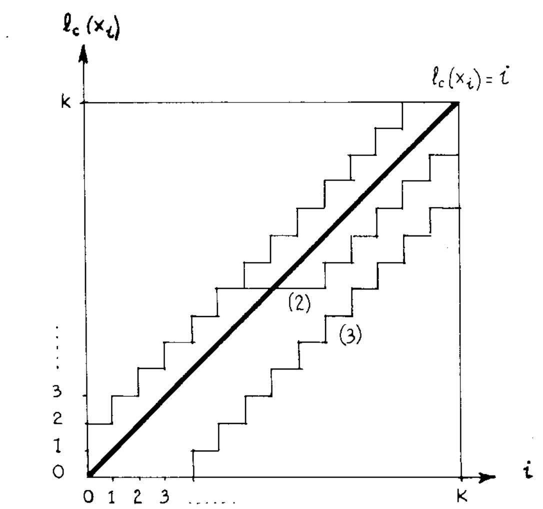

Fig. 1
Possibilities of $\ell_C(x_i)$

as then $\ell_C(x_{i_0+1})=i_0+1$ is not possible (otherwise we should choose i_0+1) and it must be $\ell_C(x_{i_0+1})=i$ (therefore $x_{i_0} \in C$).

Therefore $\varphi(C) \subseteq \mathcal{C}_{k+1}(P^{k+1})$ and $\mathcal{C}_{k+1}(P) \subseteq \mathcal{C}_{k+1}(P^{k+1})$

and

$$\underset{C \in \mathcal{C}_{k+1}(P)}{\mathrm{MIN}} \omega(C) \geq \underset{C' \in \mathcal{C}_{k+1}(P^{k+1})}{\mathrm{MIN}} \omega'(C')$$

This result, together with the one obtained in the first part of the proof, produces

$$\underset{C_{k+1} \text{ in } P}{\mathrm{MIN}} \omega(C_{k+1}) = \underset{C'_{k+1} \text{ in } P^{k+1}}{\mathrm{MIN}} \omega'(C_{k+1})$$

and the lemma is proved.

At this point, we have

$$\underset{I_k \text{ in } P}{\mathrm{MAX}} \omega(I_k) = \sum_{x \in P} \omega(x) - \underset{C_{k+1} \text{ in } P}{\mathrm{MIN}} \omega(C_{k+1}) =$$

$$= \sum_{x \in P} \omega(x) - \underset{C'_{k+1} \text{ in } P^{k+1}}{\mathrm{MIN}} \omega'(C'_{k+1})$$

Any $C' \in \mathcal{C}_{k+1}(P^{k+1})$ is a cut in P^{k+1} considered as a network, and viceversa. From the max flow = min cut theorem of Ford Fulkerson, it follows that

$$\underset{C'_{k+1} \text{ in } P^{k+1}}{\mathrm{MIN}} \omega'(C'_{k+1}) = \text{MAX FLOW } (P^{k+1})$$

and so,

$$\underset{I_K \text{ in } P}{\mathrm{MAX}} \omega(I\ ) = \sum_{x \in P} \omega(x) - \text{MAX FLOW } (P^{k+1})$$

and the Sperner-Erdös Problem is solved.

The following algorithm sums up the procedure

a) Construct P^{k+1}

b) Solve MAX FLOW on P^{k+1}. This will produce a MIN CUT C'_{k+1} and a corresponding C_{k+1} in P.

c) $I_k = P - C_{k+1}$ and I_k is the desired Maximal Weighted k-Independent Set in P.

The appendix shows an example of application.

Step a) produces a graph with k.n ($n=|P|$) - nodes. The total complexity of the algorithm will be $\Theta(kn)^{\alpha}$,α depending on the max flow algorithm which is used (α=3 for Dinic-Karzanov, α=2.33 for Galil). It is a case where ir appears pseudopolynomiality (see /5/) because the complexity dependes on k, a parameter of the input. But since the problem is trivial for $k\geq n$ the algorithm becomes polynomial.

5. REFERENCES

/1/ BERENGUER, X.,DIAZ, J.: "The Weighted - Sperner's Set Problem", Proc. of the - 9th Symp. Math. Foundations of Comp. -- Sci., Rydzyna, Poland ,1980,pp. 137-141.

/2/ BERENGUER X., DIAZ, J., HARPER, L.H.: - "A solution to the Sperner-Erdös Problem" to appear.

/3/ EVEN, S.: "The Max Flox Algorithm of -- Dinic and Karzanov: an exposition", MIT /LCS/TM-80, 1976.

/4/ GALIL, Z.: "A new algorithm for the ---- maximum flow problem", Proc. 19th IEEE ANN. Symp. on Foundations of Comp. Sci., Ann Arbor, Michigan, 1978, pp. 231-245.

/5/ GAREY, M.R., JOHNSON, D.S.: "Computers - and Intractability: A Guide to the Theory of NP-Completeness",Freeman, S. Francisco, California, 1978.

/6/ GARFINKEL, R.S., NEWHAUSER, G.L.: "Integer Programming", John Wiley, New York, 1972.

6. APPENDIX: AN EXAMPLE

Figure A-1 shows the Hasse diagram (no transitivity relations considered) of a poset P ($|P|=6$). Numbers between parenthesis indicate weights.

The Sperner-Erdös problem that is faced is to find the Maximal Weighted 2-Independent Set, I_2 in P.

Figure A-2 shows the 3-copy graph of P, P^3, including two additional vertices s(source) and t (sink) for building up the corresponding flow network. Weights on vertices are avoided (same as in P).

Heavy lines represent a Max Flow over this - network, numbers over these heavy lines are the flow values.

This yields a Max Flow $(P^3)=5$. The corresponding cut C'_3 in P^3 is $\{(1,0), (6,3), (7,3)\}$ and so the corresponding 3-chain cover in P is

$C_3 = \{1,6,7\}$. Therefore

$I_2 = P - C_3 = \{2,3,4,5\}$

and $\omega(I_2) = \sum_{x\in P} \omega(x)$ - MAX FLOW (P^3) =

13 - 5 = 8.

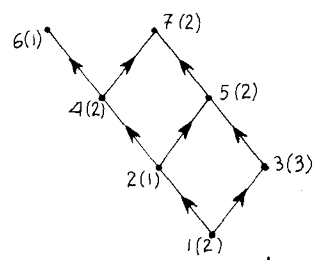

Fig. A-1

Hasse diagram of P

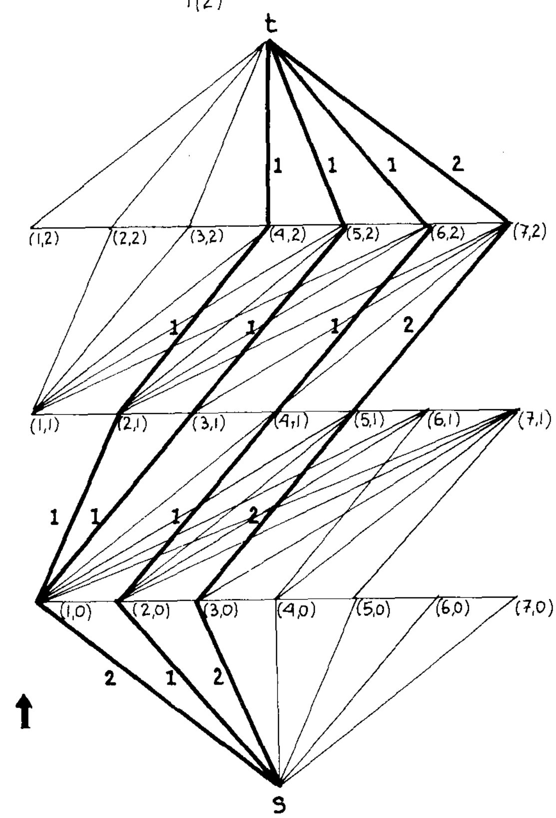

Fig. A-2

Flow network of P^3 and Max Flow