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NOTE

A SOLUTION OF THE SPERNER–ERDÖS PROBLEM

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Abstract. The purpose of this paper is to present a polynomial solution to a combinatorial optimization problem abstracted from the work of Sperner and Erdös. The solution is obtained by a series of polynomial reductions ending up with the Maxflow problem of Ford–Fulkerson.

Introduction

The purpose of this paper is to present a polynomial solution (see [5]) to a combinatorial optimization problem abstracted from the work of Sperner [9] and Erdös [1] (see [8]). The solution is obtained by a series of polynomial reductions ending up with the Maxflow problem of Ford–Fulkerson.

1. The Sperner–Erdös problem

A *k-chain* is a linearly ordered set of k elements. Given a poset P, *a Sperner set of order k* is a subset of P which contains no $(k+1)$-chains of P. Given a poset P with weight function $\omega : P \to \mathbb{R}^+$, and $k \in \mathbb{Z}^+$, the *Sperner–Erdös problem* is to find $\sigma(P, k) = \max \omega(I)$, where I ranges over all Sperner sets of order k and $\omega(I) = \sum_{x\in I} \omega(x)$. Polynomial solutions for the case $k = 1$ and $\omega(x) \equiv 1$ are mentioned in several recent publications ([6] and [10]) and seem traceable to Fulkerson's proof of Dilworth's theorem [4]. Though we were not aware of this until later, our early attempts were similar, and we found that these restricted solutions do not extend to the general problem. The case of nonconstant weights is not just a conceit, but arises naturally from that of constant weights. In [8], Harper showed how the Sperner–Erdös problem on a large poset may be reduced to that on a smaller poset and even if $\omega(x) \equiv 1$ on the former, it will generally be nonconstant on the latter.

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2. The solution

Our solution of the Sperner–Erdös problem consists of three polynomial reductions which are embodied in the following lemmas. A set $C \subseteq P$ is a *k-chain cover* if C contains at least one member of every k-chain in P. The set of all k-chain covers will be denoted by $\mathscr{C}_k(P)$.

Lemma 1.

$$\sigma(P, k) = \omega(P) - \min_{c \in \mathscr{C}_{k+1}(p)} \omega(C).$$

Proof. A set $I \subseteq P$ is a Sperner set of order k iff $C = P - I$ is a $(k+1)$-chain cover. □

$H(P)$, Hasse diagram of a poset P, is the directed graph whose vertices are the elements of P with an edge $x \to y$ if $x < y$ and $x \leq v < y$ implies $v = x$. Note that $H(P)$ determines P also. Given a weighted poset P, $P^{(k)}$ will denote a poset on k copies of P. $P^{(k)} = \{(x, i): x \in P \text{ and } i \in \mathbb{Z}, 0 \leq i \leq k\}$. Edges of $H(P^{(k)})$ will be of the form $(x, i) \to (y, i+1)$ where $x < y$ in P. The weights on $P^{(k)}$ will be defined by $\omega'(x, i) = \omega(x)$.

Example. Suppose $H(P)$ is

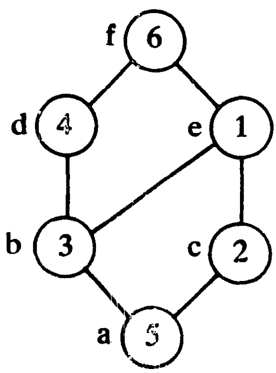

Fig. 1. $H(P)$, all edges directed upward.

Then $H(P^{(3)})$ will (essentially) be

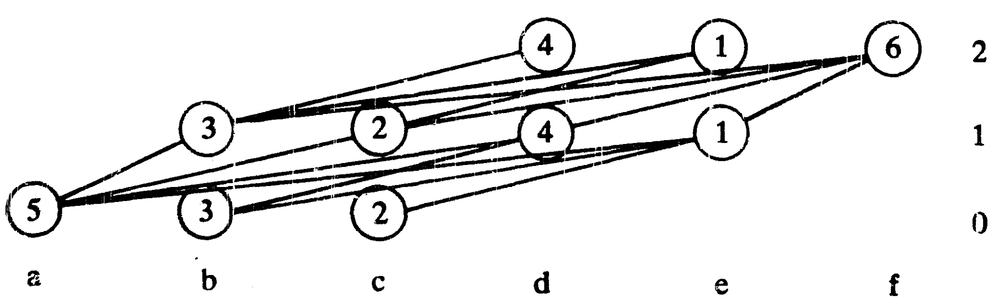

Fig. 2. $H(P^{(3)})$ with vertices not lying on any 3-path deleted.

Lemma 2.

$$\min_{C\in\mathscr{C}_{k+1}(P)} \omega(C) = \min_{C'\in\mathscr{C}_{k+1}(P^{(k+1)})} \omega'(C').$$

That is to say, the minimum $(k+1)$-chain cover problem on P and $P^{(k+1)}$ are equivalent.

Proof. Define $\rho : \mathscr{C}_{k+1}(P^{(k+1)}) \to \mathscr{C}_{k+1}(P)$ by $\rho(C') = \{x : (x, i) \in C'$ for some $i\}$. Actually we must show that $\rho(C') \in \mathscr{C}_{k+1}(P)$, but this is not difficult: If $x_0 < x_1 < \cdots < x_k$ is a $(k+1)$-chain in P, then $(x_0, 0) < (x_1, 1) < \cdots < (x_k, k)$ is a $(k+1)$-chain in $P^{(k+1)}$. Since $C' \in \mathscr{C}_{k+1}(P^{(k+1)})$, there is an i such that $(x_i, i) \in C'$, but then $x_i \in \rho(C')$. Clearly $\omega(\rho(C')) \leq \omega'(C')$, so $\min_{C\in\mathscr{C}_{k+1}(P)} \omega(C) \leq \min_{C'\in\mathscr{C}_{k+1}(P^{(k+1)})} \omega'(C')$.

To show the opposite inequality we use the same strategy but the going is more difficult: We wish to define a function $\varphi : \mathscr{C}_{k+1}(P) \to \mathscr{C}_{k+1}(P^{(k+1)})$. Given $C \in \mathscr{C}_{k+1}(P)$ and $x \in P$ let

$$l_C(x) = \max\{j : \exists x_0 < x_1 < \cdots < x_{j-1} < x \text{ and } x_i \in P - C \text{ for } 0 \leq i < j\},$$

i.e. $l_C(x)$ is the length of the longest chain in the (k-independent set) $P - C$, which leads up to (but does not include) x. Define φ then by

$$\varphi(C) = \{(x, l_C(x)) : x \in C\}.$$

Again, we must show that $\varphi(C) \in \mathscr{C}_{k+1}(P^{(k+1)})$ assuming that $C \in \mathscr{C}_{k+1}(P)$. If $L' = \{(x_0, 0)(x_1, 1), \ldots, (x_k, k)\}$ is a $(k+1)$-chain in $P^{(k+1)}$, then $L = \{x_0, x_1, \ldots, x_k\}$ is a $(k+1)$-chain in P. *We must show then that there exists $i_0 \in \mathbb{Z}^+$, $0 \leq i_0 \leq k$, such that $x_{i_0} \in C$ and $l_C(x_{i_0}) = i_0$ (for then $(x_{i_0}, i_0) \in \varphi(C)$ by definition of φ).* To see that such an i_0 exists, note that

(i) $l_C(x_0) \geq 0$,

(ii) for all i $l_C(x_{i+1}) \geq l_C(x_i)$ and $l_C(x_{i+1}) = l_C(x_i)$ implies $x_i \in C$,

(iii) $l_C(x_k) \leq k$ and $l_C(x_k) = k$ implies $x_k \in C$.

If we let $f_{C,L}(i) = l_C(x_i) - i + 1$, then corresponding to the above properties of l_C we have

(i′) $f_{C,L}(0) > 0$,

(ii′) for all i $f_{C,L}(i+1) \geq f_{C,L}(i) - 1$ and $f_{C,L}(i+1) = f_{C,L}(i) - 1$ implies $x_i \in C$,

(iii′) either $l_C(x_k) = k$ and $x_k \in C$ or $f_{C,L}(k) \leq 0$.

Thus either $l_C(x_k) = k$ and $i_0 = k$ or the set $\{i : f_{C,L}(i) = 0\}$ is non-empty (this follows from properties (i′), (ii′) and (iii′) by a discrete analogue of the intermediate value theorem). In the latter case let $i_0 = \min\{i : f_{C,L}(i) = \} - 1$, and it is easily seen that i_0 has the desired properties. Therefore the range of φ is contained in $\mathscr{C}_{k+1}(P^{(k+1)})$ and since $\omega'(\varphi(C)) = \omega(C)$, we have $\min_{C\in\mathscr{C}_{k+1}(P)} \omega(C) \geq \min_{C'\in\mathscr{C}_{k+1}(P^{(k+1)})} \omega'(C')$. Thus Lemma 2 is proved. □

Consider the Hasse diagram of $P^{(k+1)}$ as a network in the sense of Ford–Fulkerson [3], with the elements $(x, 0)$ as "sources", the elements (x, k) as "sinks", infinite capacity on all edges and capacity $\omega'(x, i) = \omega(x)$ on vertex (x, i).

Lemma 3.

$$\min_{C' \in \mathscr{C}_{k+1}(P^{(k+1)})} \omega'(C') = \text{maxflow}\, H((P^{(k+1)})).$$

Proof. Any $C' \in \mathscr{C}_{k+1}(P^{(k+1)})$ is a cut in the Hasse diagram of $P^{(k+1)}$ considered as a network, and vice versa. The lemma then follows from the maxflow = mincut theorem of Ford–Fulkerson. □

Putting Lemmas 1, 2 and 3 together we have our main result.

Theorem. *For any weighted poset P and* $k \in \mathbb{Z}^+$

$$\sigma(P, k) = \omega(P) - \text{maxflow}\, H((P^{(k+1)})).$$

Example. In the diagram of Fig. 2, the following is a maximum flow:

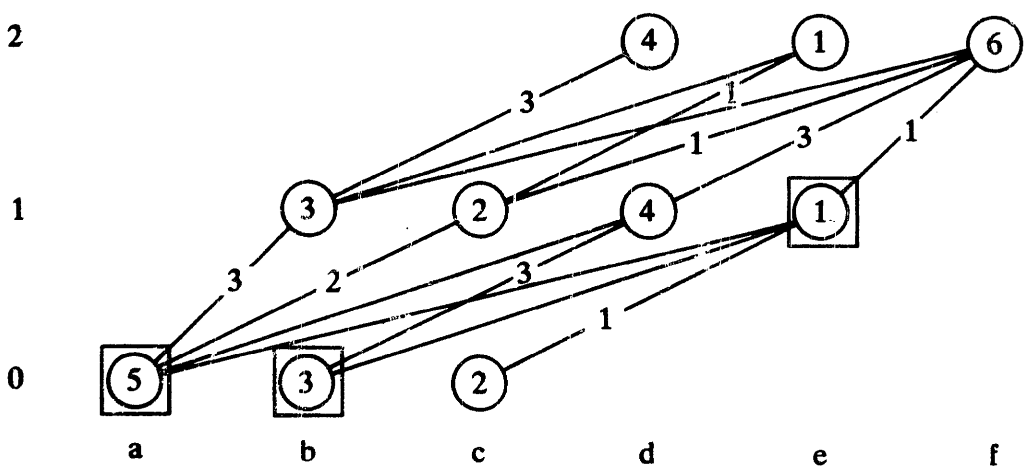

Fig. 3. Edges without numbers have flow 0.

The vertices in the squares are the corresponding minimum cut. Thus $\sigma(P, 2) = 21 - 9 = 12$ and $\{c, d, f\}$ is the maximum weight Sperner set of order 2 in P of Fig. 1.

Using the Dinic–Karzanov algorithm [2], $\sigma(P, k)$ may be computed in $O((kn)^3))$ time, where $n = |P|$. Also I, the Sperner set of order k for which $\omega(I) = \sigma(P, k)$ may be found in $O((kn)^3)$ time. This appears to be only pseudopolynomial in k (see [5]), but since the problem is trivial for $k \geqslant n$, we obtain a bound of $O(n^6)$.

3. Afterword

Following the circulation of this paper in preprint form, M. Saks pointed out to us some additional connections with previous publications: In [12] Saks showed that for any poset, P, and $k \in \mathbb{Z}^+$, the Sperner–Erdös problem for P and k (wth $\omega \equiv 1$) is equivalent to that for $P \times [k]$ and $k' = 1$, $[k]$ being a chain on k elements and "$\times$" denoting poset product. This result may be extended to arbitrary positive real weights by an argument similar to that for Lemma 2. The case $k = 1$ of the Sperner–Erdös problem is considerably simpler and Saks believes that he has seen our reduction in that case before (at least for $\omega = 1$), though he does not know whom to attribute it to. This series of reductions also gives a polynomial solution to the Sperner–Erdös problem, but we believe that ours is worthy of being added to the literature because of its completeness and lower algorithmic complexity.

$P^{(k)}$, again with $\omega \equiv 1$ only, was defined by S.V. Fomin [11] although the use to which he puts it seems different from ours and his result appears not to extend to arbitrary weights.

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